TRANSITIVE CIRCLE EXCHANGE MAPS WITH FLIPS

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ABSTRACT. We study the existence of transitive exchange maps with flips defined on the unit circle S^1. We provide a complete answer to the question of whether there exists a transitive exchange map of S^1 defined on n subintervals and having f flips.

1. INTRODUCTION

Circle exchange transformations (CETs) and interval exchange transformations (IETs) form a fundamental class of objects studied in measurable dynamics, noted for their simple definition yet rich variety of behaviours (see [16]). They arise naturally in the study of polygonal billiards (see [1]), measured foliations (see [2, 9]) and flat surfaces (see [17, 18]).

Moreover, transitive CETs arise in the study of surface flows, as a factor of the Poincaré return map to a transversally embedded circle. The much-studied orientation-preserving CETs codify the non-trivial recurrence of flows on orientable surfaces (see [3, 6]). Advancement in the understanding of flows on *non-orientable* surfaces, such as an answer for the open question about the C^r-density, $r \geq 2$, of the Morse–Smale vector fields on non-orientable surfaces (see [5, 12]), requires study of the non-trivial recurrence in CETs that are not orientation-preserving, but rather reverse orientation on one or more subintervals, called "flips".

Generically, exchange maps with flips have periodic points and so are not transitive (see [11]). Nevertheless, for every even $n \geq 4$ there exist transitive CETs with flips defined on n subintervals (see [4, 10]). In this article we extend this result: we determine for which pairs $(n, f) \in \mathbb{N}^2$ there exists a transitive CET defined on n subintervals and having f flips.

We look for the least number of subintervals and flips required for a CET to be transitive. As remarked by Keane [7], every CET with flips defined on two subintervals has a periodic point, and thus is not transitive. Thus we begin by looking at CETs defined on three subintervals. The examples presented in this paper then work as building blocks of transitive, uniquely ergodic CETs having arbitrary numbers of subintervals and flips.

2. STATEMENT OF THE RESULTS

Let $S^1 := [0, 1]/(0 \sim 1)$ be a circle endowed with the orientation induced via the natural projection $[0, 1] \to S^1$. We denote points of S^1 as if they were points of $[0, 1]$. Let $I_1, I_2, \ldots, I_n$ be a collection of pairwise disjoint open subintervals of S^1 whose closure covers S^1. A *circle exchange transformation of n subintervals*, shortly

an n-CET, is an injective map $T : \bigcup_{i=1}^{n} I_i \subset S^1 \to S^1$ which acts as an isometry on each interval I_i, $1 \le i \le n$, and whose domain $\text{Dom}(T) = \bigcup_{i=1}^{n} I_i$ cannot be continuously extended to a bigger open subset of S^1. A CET is called *oriented* if it preserves the orientation of all the domain subintervals, otherwise, if it reverses the orientation of one or more domain subintervals, called *flips*, it is said to be a CET *with flips*. By replacing the circle S^1 by a closed interval $[a, b]$, $0 \le a < b$, we get the definition of *interval exchange transformation*. We shall use the expression *exchange transformation* to refer to both CETs and IETs. Notice that an n-CET (respectively, an n-IET) has necessarily n (respectively, $n-1$) discontinuity points.

Let $\Gamma \in \{S^1, [a, b]\}$ and let $T : \Gamma \to \Gamma$ be an exchange map. The orbit of $p \in \Gamma$ is the set

$$O(p) = \{T^n(p) \mid n \in \mathbb{Z} \text{ and } p \in \text{Dom}(T^n)\}.$$

We say that T is *transitive* if there exists an orbit of T that is dense in Γ. We say that the orbit of $p \in \Gamma$ is *finite* if the set $O(p)$ has finite cardinality. Accordingly, a point $p \in \Gamma - (\text{Dom}(T) \cup \text{Dom}(T^{-1}))$ has the finite orbit $O(p) = \{p\}$. A transitive exchange transformation is *minimal* if it has no finite orbits. The exchange transformation T is called *uniquely ergodic* if the Lebesgue measure on Γ is the only (up to scalar multiple) Borel measure invariant by T. Notice that for exchange transformations, unique ergodicity implies transitivity.

The results of this paper may be combined to make the following statement.

Main Theorem. *Given $n \ge f \ge 1$, there exists a transitive CET of n subintervals with f flips if and only if $n + f \ge 5$.*

We denote by $\mathcal{C}(n, f)$ the set of transitive CETs of n subintervals with f flips. Since $C(2, f)$ is empty for $f = 1, 2$ (see [7]), it suffices to consider the case of $n \ge 3$. Thus the above result follows from the three statements:

(i) $\mathcal{C}(3, 1) = \emptyset$ (Theorem 3.1);
(ii) $\mathcal{C}(3, f) \ne \emptyset$ for $f = 2, 3$ (Theorems 6.1 and 7.1);
(iii) $\mathcal{C}(n, f) \ne \emptyset$ for every $n \ge 4$ and $1 \le f \le n$ (Theorem 9.1).

3. 3-CET WITH EXACTLY ONE FLIP

The main result of this section is the following theorem.

Theorem 3.1. *Every 3-CET with exactly one flip has a periodic point and thus $\mathcal{C}(3, 1) = \emptyset$.*

The proof of the Theorem 3.1 follows from the lemmas below.

Definition 3.2. Given $p \in S^1$, we identify $S^1 - \{p\}$ with a subinterval of $\mathbb{R}$ endowed with the total order induced by the cyclic order. Accordingly, given a subinterval J of $S^1 - \{p\}$ and $\delta \in \mathbb{R}$, with $|\delta|$ small enough, we may set $J + \delta = \{x + \delta : x \in J\}$ to be the shift of J by δ. Given subintervals J_1, J_2 of $S^1 - \{p\}$, we say that $J_1 < J_2$ if $x < y$ for all $(x, y) \in J_1 \times J_2$. Analogously, if $\{J_i\}_{i=1}^{r}$, $r \ge 3$, is a collection of subintervals of S^1, we say that $J_1 < J_2 < \ldots < J_r$ if every $(x_i)_{i=1}^{r} \in \Pi_{i=1}^{r} J_i$ is cyclically ordered.

Throughout this section, we let $T : S^1 \to S^1$ be a 3-CET with one flip defined on the domain $\text{Dom}(T) = S^1 - \{a_0, a_1, a_2, a_3 = a_0\}$, where $0 = a_0 < a_1 < a_2 < a_3 = 1$ and $I_i = (a_{i-1}, a_i)$, $i = 1, 2, 3$. Without loss of generality, we may assume that I_1 is the flip of T and that $T(I_1) < T(I_3) < T(I_2)$ (otherwise T would be a 2-CET with

flips). We denote by T^0 the identity map on S^1 and we suppose by contradiction that T has no periodic points. We say that $p \in S^1$ is a *flipped* periodic point of T if for some $n \in \mathbb{N}$, $T^n(p) = p$ and $DT^n(p) = -1$.

Lemma 3.3. *There exists $N \geq 2$ such that $T^n(I_1) \cap I_1 = \emptyset$ for all $1 \leq n \leq N - 1$, and $T^N(I_1) \cap I_1 \neq \emptyset$. Furthermore, $\{T^n(I_1)\}_{n=0}^{N-1}$ are pairwise disjoint.*

Proof. The proof follows by induction. Firstly note that $T(I_1) \cap I_1 = \emptyset$, otherwise T would have a flipped periodic point in I_1. Now assume that for some $k \in \{1, 2, \dots\}$, $\{T^n(I_1)\}_{n=0}^{k}$ are pairwise disjoint and that $T^{k+1}(I_1) \cap I_1 = \emptyset$. We claim that $\{T^n(I_1)\}_{n=0}^{k+1}$ are pairwise disjoint. In fact, if this was not the case then there would exist integers $1 \leq m_1 < m_2 \leq k + 1$ such that $T^{m_2}(I_1) \cap T^{m_1}(I_1) \neq \emptyset$. Hence, there would exist $x_2 \in T^{m_2-1}(I_1)$ and $x_1 \in T^{m_1-1}(I_1)$ such that $T(x_2) = T(x_1)$. By hypothesis, $T^{m_2-1}(I_1) \cap T^{m_1-1}(I_1) = \emptyset$ and so $x_2 \neq x_1$, which contradicts the injectivity of T. This proves the claim. To finish the proof, note that if $\{T^n(I_1)\}_{n=0}^{k}$ are pairwise disjoint then $\mu(\bigcup_{n=0}^{k} T^n(I_1)) = (k + 1)\mu(I_1)$, where μ stands for the Lebesgue measure on S^1, which is invariant by T. Hence, for some $k > 0$, $T^k(I_1) \cap I_1 \neq \emptyset$. We set N to be the least k with such property. $\square$

By Lemma 3.3, $D = \left(\bigcup_{n=0}^{N-1} T^n(I_1)\right) \cap \{a_2\}$ has at most one point, i.e., a_2 belongs to at most one interval from $\bigcup_{n=0}^{N-1} T^n(I_1)$. Hence, there are two cases to consider: either $D = \emptyset$ or D is a one-point set. Assume first that $D = \emptyset$. Then $T^n(I_1) \cap \{a_0, a_1, a_2\} = \emptyset$ for all $0 \leq n \leq N - 1$. Consequently, I_1 and $T^N(I_1)$ are non–disjoint intervals with opposed orientations, and so T has a flipped periodic point in I_1, which contradicts the initial assumption that T has no periodic points. Therefore, we may assume that $D \neq \emptyset$. Hence, for some $1 \leq n_0 \leq N - 1$, $T^{n_0}(I_1) \cap \{a_2\} \neq \emptyset$. Let $d \in I_1$ be such $T^{n_0}(d) = a_2$ and let $I_1 - \{d\} = I_\ell \cup I_r$, where $I_\ell < I_r$ in $S^1 - \{a_2\}$. Keeping these assumptions, we have the following lemma.

Lemma 3.4. *Either $a_0 \in T^N(I_\ell)$ or $a_1 \in T^N(I_r)$.*

Proof. Since $T^N(I_1) \cap I_1 \neq \emptyset$, we have that either $T^N(I_\ell) \cap I_1 \neq \emptyset$ or $T^N(I_r) \cap I_1 \neq \emptyset$. We first assume that $T^N(I_r) \cap I_1 \neq \emptyset$. Note that each $J_n = T^n(I_r)$, $1 \leq n \leq N$, is an interval. For each $1 \leq n \leq N$, let $k_n \geq 0$ and $1 \leq q_n \leq n_0$ be integers such that $n = k_n n_0 + q_n$. We claim that for all $1 \leq n \leq N$,

$$J_n \subset S^1 - \{a_0\} \quad \text{and} \quad J_n = J_{q_n} - k_n \mu(I_r).$$

The claim holds trivially for all $1 \leq n \leq n_0$. Since $T(I_2) < T(I_1) < T(I_3)$ and $J_{n_0} = (c_{n_0}, a_2) \subset I_2$, we have that $J_{n_0+1} = J_1 - \mu(I_r)$.

Now, given $n_0 + 1 < n \leq N - 1$, assume that the claim holds for all $1 \leq k \leq n$. Let us prove that it also holds for $n + 1$. By the hypothesis of induction, $J_{n-n_0} = J_{q_n} - (k_n - 1)\mu(I_r)$ and $J_n = J_{q_n} - k_n \mu(I_r)$. Hence, $J_n = J_{n-n_0} - \mu(I_r)$. Consequently (see Figure 1),

$$(3.1) \quad J_{n+1} = T(J_n) = T\big(J_{n-n_0} - \mu(I_r)\big) = T(J_{n-n_0}) - \mu(I_r) = J_{n-n_0+1} - \mu(I_r).$$

Using the hypothesis of induction once more, we obtain that

$$(3.2) \qquad J_{n-n_0+1} = \begin{cases} J_{q_n+1} - (k_n - 1)\mu(I_r) & \text{if } 1 \leq q_n < n_0; \\ J_1 - k_n \mu(I_r) & \text{if } q_n = n_0. \end{cases}$$

Putting together the equations (3.1) and (3.2) we reach

$$J_{n+1} = \begin{cases} J_{q_n+1} - k_n\mu(I_r) = J_{q_{n+1}} - k_{n+1}\mu(I_r) & \text{if } 1 \le q_n < n_0; \\ J_1 - (k_n + 1) = J_{q_{n+1}} - k_{n+1}\mu(I_r) & \text{if } q_n = n_0. \end{cases}$$
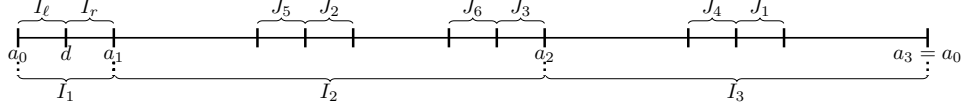
which proves the claim.

FIGURE 1. The iterates of I_r under the map T.

Hence, we have that either $T^N(I_r) = I_r$ (which contradicts the assumption that T has no periodic points) or $a_1 \in T^N(I_r)$. The proof of the other case is similar. □

Proof of Theorem 3.1. It follows at once from Lemma 3.4. In fact, the orientation of $T^N(I_\ell)$ and $T^N(I_r)$ is opposed to the orientation of I_ℓ and I_r. By Lemma 3.4, either $T^N(I_\ell) \cap I_\ell \ne \emptyset$ or $T^N(I_r) \cap I_r \ne \emptyset$, which contradicts the initial assumption that T has no periodic points. This finishes the proof because if a CET has a periodic point then it has a whole interval of periodic points, and as such cannot be transitive. □

4. RAUZY INDUCTION FOR IETs WITH FLIPS

The Rauzy induction is a renormalisation algorithm for interval exchange transformations [13]. Using this method, Veech [14] proved that almost all oriented IETs are uniquely ergodic. For explanations of the oriented IET case, see [14]. Here we present the Rauzy procedure for IETs with flips, see [11]. We first introduce coordinates into the space of interval exchange transformations.

Let S_n be the permutation group on $\{1, 2, \ldots, n\}$. We may represent any permutation $\pi \in S_n$ by the n-tuple

$$\pi = (\pi(1), \pi(2), \ldots, \pi(n)) = (\pi_1, \pi_2, \ldots, \pi_n).$$

We shall denote by S_n^* the subset of S_n made up by the irreducible permutations: $\pi \in S_n^*$ if $\pi(\{1, 2, \ldots, k\}) = \{1, 2, \ldots, k\}$ implies $k = n$. Given $\theta \in \{-1, 1\}^n$ and $\pi \in S_n$, we let $\theta\pi = (\theta_1\pi_1, \theta_2\pi_2, \ldots, \theta_n\pi_n)$ be the signed permutation obtained by componentwise multiplication of θ and π. We shall denote by Σ_n the set of signed permutations

$$\Sigma_n = \{\theta\,\pi \mid \theta \in \{-1, 1\}^n \text{ and } \pi \in S_n\}.$$

We say that a signed permutation $p = \theta\,\pi$ is *irreducible* if π is irreducible. We shall denote by Σ_n^* the set of irreducible signed permutations on n symbols.

To each n-IET $T : [0, \ell] \to [0, \ell]$, $\ell > 0$, there corresponds a unique point (λ, p) in the space $\Lambda_n \times \Sigma_n$, where

$$\Lambda_n = \{\lambda = (\lambda_1, \ldots, \lambda_n) \mid \lambda_i > 0, \forall i\}$$

is the positive cone. We say that $(I_1, \ldots, I_n)$ is a *partition* of $[0, \ell]$ if there exist real numbers $0 = a_0 < a_1 < \cdots < a_n = \ell$ such that $I_i = (a_{i-1}, a_i)$, $i = 1, 2, \ldots, n$. We let $|J| = \sup\{y - x \mid x, y \in J\}$. Given an n-IET $T : [0, \ell] \to [0, \ell]$, we denote by $(I_1, \ldots, I_n)$ and $(J_1, \ldots, J_n)$, respectively, the domain partition and the range partition defined by T. We assign the data $(\lambda, p) \in \Lambda_n \times \Sigma_n$ to T in the following

way: for each $i = 1, 2, \ldots, n$, $\lambda_i = |I_i|$, and $p = \theta\,\pi$, where $\pi(i)$ is such that $J_{\pi(i)} = T(I_i)$, and $\theta_i = 1$ (respectively $\theta_i = -1$) if T preserves (respectively reverses) the orientation of I_i. Conversely, to each $(\lambda, p) \in \Lambda_n \times \Sigma_n$, there corresponds a unique IET of $k \leq n$ subintervals that we shall denote by $T = \mathcal{E}(\lambda, p)$.

Note that if $p \in \Sigma_n^*$ is irreducible then any IET $T = \mathcal{E}(\lambda, p)$ associated to the signed permutation p can be decomposed into several IETs with smaller number of subintervals each. Since we are interested in transitive exchange transformations, we shall only consider irreducible permutations.

The *Rauzy induction* is the operator on the space of IETs that associates to $T = \mathcal{E}(\lambda, p)$ the interval exchange transformation $T' = \mathcal{E}(\lambda', p')$ which is the Poincaré first return map induced by T on the subinterval $I(\lambda) = [0, \nu]$, where $|\lambda| = \sum \lambda_i$, $p = \theta\,\pi$ and

$$\nu = \begin{cases} |\lambda| - \lambda_{\pi^{-1}(n)}, & \text{if } \lambda_{\pi^{-1}(n)} < \lambda_n \\ |\lambda| - \lambda_n, & \text{if } \lambda_{\pi^{-1}(n)} > \lambda_n \end{cases}.$$

In this way, we obtain the map $\mathcal{T}_0 : \Lambda_n \times \Sigma_n \to \Lambda_n \times \Sigma_n$, where

$$\mathcal{T}_0(\lambda, p) = (\lambda', p')$$

and an associated projective map

$$\mathcal{T} : \Delta_{n-1} \times \Sigma_n \to \Delta_{n-1} \times \Sigma_n, \quad \Delta_{n-1} = \{\lambda \in \Lambda_n \mid |\lambda| = 1\},$$

where

$$\mathcal{T}(\lambda, p) = \left(\frac{\lambda'}{|\lambda'|}, p' \right).$$

Notice that the operators $\mathcal{T}_0$ and $\mathcal{T}$ are defined only on the sets

$$\begin{aligned} \text{Dom}(\mathcal{T}_0) &= \{(\lambda, \theta\,\pi) \in \Lambda_n \times \Sigma_n \mid \lambda_n \neq \lambda_{\pi^{-1}(n)}\}, \\ \text{Dom}(\mathcal{T}) &= \{(\lambda, \theta\,\pi) \in \Delta_{n-1} \times \Sigma_n \mid \lambda_n \neq \lambda_{\pi^{-1}(n)}\}. \end{aligned}$$

The operator $\mathcal{T}$ may be written in the form

$$\mathcal{T}(\lambda, \theta\,\pi) = \begin{cases} \left(\frac{M_a(\theta\,\pi)^{-1}\lambda}{|M_a(\theta\,\pi)^{-1}\lambda|}, a(\theta\,\pi) \right) & \text{if} \quad \lambda_{\pi^{-1}(n)} < \lambda_n \\ \left(\frac{M_b(\theta\,\pi)^{-1}\lambda}{|M_b(\theta\,\pi)^{-1}\lambda|}, b(\theta\,\pi) \right) & \text{if} \quad \lambda_{\pi^{-1}(n)} > \lambda_n \end{cases},$$

where the transition matrices $M_a(\theta\,\pi), M_b(\theta\,\pi) \in \text{SL}_n(\mathbb{Z})$ and the transition maps $a, b : \Sigma_n \to \Sigma_n$ are described below. For $i, j = 1, \ldots, n$, denote by E_{ij} the $n \times n$ matrix of which the (i, j)th element is equal to 1, and all the others elements are equal to 0. Let E be the $n \times n$ identity matrix. The transition matrices are defined by:

$$\begin{aligned} M_a(\theta\,\pi) &= E + E_{n, \pi^{-1}(n)}, \\ M_b(\theta\,\pi) &= \sum_{i=1}^{\pi^{-1}(n)} E_{i,i} + E_{n, s(\theta\,\pi)} + \sum_{i=\pi^{-1}(n)}^{n-1} E_{i,i+1}, \end{aligned}$$

where $s(\theta\,\pi) = \pi^{-1}(n) + (1 + \theta_{\pi^{-1}(n)})/2$.

We now define the transition maps. When $\theta_n = +1$ the transition map a is defined by

$$a(\theta\,\pi)_i = \begin{cases} \theta_i \pi_i, & \pi_i \leq \pi_n, \\ \theta_i(\pi_n + 1), & \pi_i = n, \\ \theta_i(\pi_i + 1), & \text{otherwise,} \end{cases}$$

and when $\theta_n = -1$, we have

$$a(\theta\,\pi)_i = \begin{cases} \theta_i\pi_i, & \pi_i \leq \pi_n - 1, \\ -\theta_i\pi_n, & \pi_i = n, \\ \theta_i(\pi_i + 1), & \text{otherwise.} \end{cases}$$

The transition map b when $\theta_{\pi^{-1}(n)} = +1$ is defined by

$$b(\theta\,\pi)_i = \begin{cases} \theta_i\pi_i, & i \leq \pi^{-1}(n), \\ \theta_n\pi_n, & i = \pi^{-1}(n) + 1, \\ \theta_{i-1}\pi_{i-1}, & \text{otherwise,} \end{cases}$$

and when $\theta_{\pi^{-1}(n)} = -1$ by

$$b(\theta\,\pi)_i = \begin{cases} \theta_i\pi_i, & i \leq \pi^{-1}(n) - 1, \\ -\theta_n\pi_n, & i = \pi^{-1}(n), \\ \theta_{i-1}\pi_{i-1}, & \text{otherwise.} \end{cases}$$

5. Renormalisable interval exchange maps

We shall denote by $\mathcal{G}_n$ the graph with vertices in Σ_n^* and edges in $\mathbb{E}_n = \mathbb{E}_n^a \cup \mathbb{E}_n^b$, where

$$\begin{aligned} \mathbb{E}_n^a &= \{(p, q) \in \Sigma_n^* \times \Sigma_n^* \mid q = a(p)\} \\ \mathbb{E}_n^b &= \{(p, q) \in \Sigma_n^* \times \Sigma_n^* \mid q = b(p)\} \end{aligned}$$

and $a, b : \Sigma_n \to \Sigma_n$ are the transitions maps defined previously. Since $\mathbb{E}_n^a \cap \mathbb{E}_n^b = \emptyset$, there is a function $\tau : \mathbb{E}_n \to \{a, b\}$ which assigns to each edge e of $\mathcal{G}_n$ its type: a (if $e \in \mathbb{E}_n^a$) or b (if $e \in \mathbb{E}_n^b$). Let N be a positive integer. A (directed) *finite path* of $\mathcal{G}_n$ is a sequence $\{p^{(i)}\}_{i=0}^N$ such that $(p^{(i-1)}, p^{(i)}) \in \mathbb{E}_n$ for all $1 \leq i \leq N$. To each finite path $\{p^{(i)}\}_{i=0}^N$, there are associated a *type itinerary* $\{t_i\}_{i=1}^N$, defined by $t_i = \tau(p^{(i-1)}, p^{(i)})$ for $i = 1, 2, \ldots, N$, and a *matrix itinerary* $\{M_\gamma^{(i)}\}_{i=1}^N$ consisting of the product of transition matrices

$$M_\gamma^{(i)} = M_{t_1}(p^{(0)}) \cdot M_{t_2}(p^{(1)}) \cdots M_{t_i}(p^{(i-1)}).$$

By replacing the set $\{0, 1, \ldots, N\}$ by $\mathbb{N} \cup \{0\}$, we obtain the definition of *infinite path*. In the same way as above, we get the definition of type itinerary $\{t_i\}_{i=1}^\infty$ and of matrix itinerary $\{M_\gamma^{(i)}\}_{i=1}^\infty$ associated to an infinite path $\{p^{(i)}\}_{i=0}^\infty$. We say that a finite path $\{p^{(i)}\}_{i=0}^N$ is a *cycle* if $p^{(N)} = p^{(0)}$. To every cycle $\gamma = \{p^{(i)}\}_{i=0}^N$ there are associated the *product matrix* $M_\gamma^{(N)}$ and the infinite path $\{p^{(i)}\}_{i=0}^\infty$ where $p^{(j)} := p^{(j \bmod N)}$ if $j > N$. We call such infinite path a *periodic path of period N*.

We say that a matrix $A \in \mathrm{SL}_n(\mathbb{Z})$ is *eventually positive* if A^n is a positive matrix for some positive integer n. A cycle $\gamma = \{p^{(i)}\}_{i=0}^N$ is *special* if the associated product matrix $M = M_\gamma^{(N)}$ is eventually positive. A path $\{p^{(i)}\}_{i=0}^N$ is a *special path* if $p^{(N)}$ belongs to a special cycle. A periodic path $\gamma = \{p^{(i)}\}_{i=0}^\infty$ of period N is *special* if $M_\gamma^{(N)}$ is eventually positive.

We say that two cycles $\{p^{(i)}\}_{i=0}^N$ and $\{q^{(j)}\}_{j=0}^N$ are *shift equivalent* if there exists $r \in \mathbb{Z}$ such that $q^{(i)} = p^{((i+r) \bmod N)}$ for all $i \in \{0, 1, \ldots, N\}$. The property of being a special cycle is invariant under shift equivalence.

We say that $T = \mathcal{E}(\lambda^{(0)}, p^{(0)})$ is *infinitely renormalisable* if there exists a sequence of pairs $\{(\lambda^{(i)}, p^{(i)})\}_{i=0}^\infty$ in $\mathrm{Dom}(\mathcal{T}_0) \cap (\Lambda_n \times \Sigma_n^*)$ such that for all $i > 0$,

$$(\lambda^{(i)}, p^{(i)}) = \mathcal{T}_0(\lambda^{(i-1)}, p^{(i-1)}).$$

In this case, the infinite path $\{p^{(i)}\}_{i=0}^{\infty}$ is called the *path corresponding to T*. Not every infinite path in $\mathcal{G}_n$ corresponds to an infinitely renormalisable IET. Nevertheless, special periodic paths (and so special cycles) always correspond to infinitely renormalisable IETs.

Given an infinite path $\gamma = \{p^{(i)}\}_{i=0}^{\infty}$ of $\mathcal{G}_n$, let

$$C(\gamma) = \bigcap_{i=1}^{\infty} M_{\gamma}^{(i)} \Lambda_n.$$

A vector λ of Λ_n is in $C(\gamma)$ if and only if $\mathcal{E}(\lambda, p^{(0)})$ is infinitely renormalisable and its corresponding path is γ. As in the oriented case (see [16]), if $T = \mathcal{E}(\lambda, p^{(0)})$ is infinitely renormalisable, then it is minimal. The dimension of $\mathcal{C}(\gamma)$ is strictly less than n, and $C(\gamma)$ is linearly isomorphic to $\mathcal{M}(T)$, the set of invariant measures of T that are positive on open sets. Thus T is uniquely ergodic if and only if $\mathcal{C}(\gamma)$ is a half-line.

We now explain how to construct infinitely renormalisable IETs with flips. Let $\{p^{(i)}\}_{i=0}^{N}$ be a special cycle and let $\gamma = \{p^{(i)}\}_{i=0}^{\infty}$ be the corresponding special periodic path. By hypothesis the matrix $M = M_{\gamma}^{(N)}$ is eventually positive. The Perron–Frobenius Theorem asserts that M has a positive, simple eigenvalue σ of maximum modulus, and a unique eigenvector $\lambda \in \Delta_{n-1}$ such that $M\lambda = \sigma\lambda$. Thus

$$\lambda \in \mathcal{C}(\gamma) = \bigcap_{i=1}^{\infty} M_{\gamma}^{(i)} \Lambda_n \subset \bigcap_{j=1}^{\infty} M^j \Lambda_n = \text{span}(\lambda).$$

Hence, taking $\lambda^{(0)} = \lambda$, we have that the domain of $\mathcal{E}(\mathcal{T}_0^N(\lambda^{(0)}, p^{(0)}))$ is $[0, 1/\sigma]$, and $(\lambda^{(0)}, p^{(0)})$ is a periodic point of $\mathcal{T}$ of period N. Thus $T_0 = \mathcal{E}(\lambda^{(0)}, p^{(0)})$ is infinitely renormalisable, and therefore minimal. By the uniqueness of the positive eigendirection of M, we have that $\mathcal{C}(\gamma_0)$ is a half-line, and so T_0 is uniquely ergodic. As $(\lambda^{(0)}, p^{(0)})$ is a fixed point of the renormalisation operator $\mathcal{T}^N$, we say that the IET $T_0 = \mathcal{E}(\lambda^{(0)}, p^{(0)})$ is *self-induced*.

6. Minimal 3-CET with two flips

In this section we present a special cycle of $\mathcal{G}_4$, from which we construct a self-induced interval exchange transformation of four subintervals with two flips. We also make use of this cycle in each of the subsequent sections.

Given a pair $(n, f) \in \mathbb{N}^2$, we let $\mathcal{I}(n, f)$ (respectively, $\mathcal{C}(n, f)$) denote the set of transitive IETs (respectively, CETs) of n subintervals with f flips.

Theorem 6.1. *There exists a minimal, uniquely ergodic 4-CET with two flips. In particular, $\mathcal{I}(4, 2) \neq \emptyset$ and $\mathcal{C}(3, 2) \neq \emptyset$.*

Proof. Let $\gamma_1 = \{p^{(i)}\}_{i=0}^{9}$ be the cycle of $\mathcal{G}_4$ that starts at the signed permutation $p^{(0)} = (-3, +4, +1, -2)$, has type itinerary $\{a, b, a, b, b, a, b, a, b\}$ and is described in the Figure 2.

The product matrix associated to the cycle γ_1 is

$$M = M_{\gamma_1}^{(9)} = \begin{pmatrix} 2 & 2 & 3 & 2 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 \\ 1 & 2 & 3 & 3 \end{pmatrix}.$$
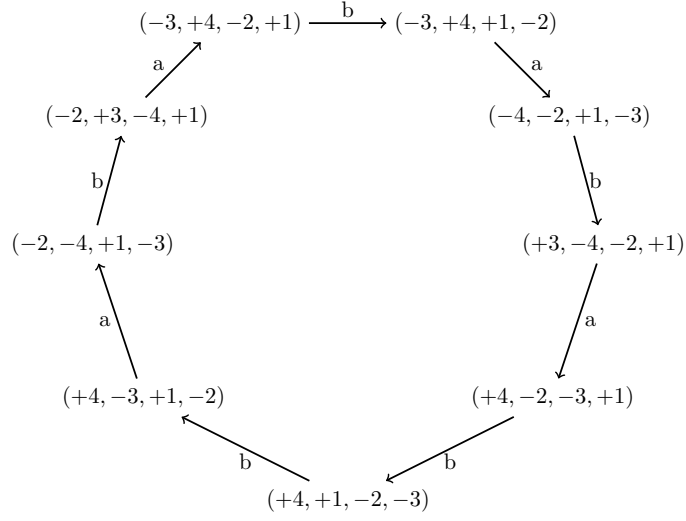
FIGURE 2. A special cycle of $\mathcal{G}_4$.

The matrix M is eventually positive and so γ_1 is a special cycle. If σ is the Perron–Frobenius eigenvalue of M and $\lambda \in \Delta_3$ is the corresponding eigenvector then the 4-IET $T_1 = \mathcal{E}(\lambda^{(0)}, p^{(0)})$ of $[0, 1]$ with $\lambda^{(0)} = \lambda$ and permutation $p^{(0)} = (-3, +4, +1, -2)$ is self-induced on the interval $[0, 1/\sigma]$ (see Figure 5). Notice that the path of $\mathcal{G}_4$ corresponding to T_1 is the infinite periodic path generated by γ_1, i.e., the path $\{p^{(i)}\}_{i=0}^{\infty}$, where $p^{(j)} := p^{(j \bmod (9))}$ for all $j > 9$.

If the endpoints of $[0, 1]$ are identified, T_1 becomes a minimal, uniquely ergodic 3-CET with two flips as required because the union $I_2 \cup \{a_2\} \cup I_3$ is mapped onto $J_4 \cup \{0 \sim 1\} \cup J_1$. $\square$

7. MINIMAL 3-CET WITH THREE FLIPS

In this section we prove the following result:

Theorem 7.1. *There exists a minimal, uniquely ergodic 4-CET with three flips. In particular, $\mathcal{I}(4, 4) \neq \emptyset$ and $\mathcal{C}(3, 3) \neq \emptyset$.*

Proof. Let $\gamma_2 = \{q^{(j)}\}_{j=0}^{4}$ be the path of $\mathcal{G}_4$ that starts at the signed permutation $q^{(0)} = (-3, -1, -4, -2)$, has type itinerary $\{a, b, b, a\}$, and is described in Figure 3. Note that γ_2 is a special path, since $q^{(4)} = (+4, -3, +1, -2)$ is a vertex in the special cycle γ_1.
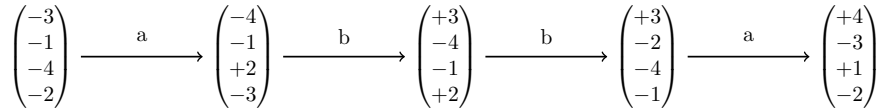
FIGURE 3. A path in $\mathcal{G}_4$ terminating on the cycle of Figure 2.

The product matrix for the path γ_2 is

$$M_{\gamma_2}^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix}.$$

Setting $\mu := M_{\gamma_2}^{(4)}(M_{\gamma_1}^{(5)})^{-1}\lambda$ yields a 4-IET $T_2 := \mathcal{E}(\mu/|\mu|, q^{(0)})$ of $[0,1]$ with four flips (see Figure 5) that is minimal and uniquely ergodic, since there is a subinterval to which the Poincaré first return map is minimal and uniquely ergodic. Identifying the endpoints of $[0,1]$, since the interval $I_2 \cup \{a_2\} \cup I_3$ is mapped onto the interval $J_4 \cup \{0 \sim 1\} \cup J_1$, we have a minimal, uniquely ergodic 3-CET with three flips. $\quad\square$

8. MINIMAL 4-CET WITH ONE FLIP

In this section we prove the following result, thus showing that Theorem 3.1 cannot be extended to n-CETs, $n > 3$.

Theorem 8.1. *There exists a minimal, uniquely ergodic 4-IET with one flip. In particular, $\mathcal{I}(4,1) \neq \emptyset$ and $\mathcal{C}(4,1) \neq \emptyset$.*

Proof. We define $\gamma_3 = \{r^{(k)}\}_{k=0}^4$ to be the path in $\mathcal{G}_4$ that starts at $r^{(0)} = (-4, +1, +3, +2)$, has type itinerary $\{b, b, a, b\}$ and finishes at the vertex $r^{(4)} = (-3, -1, -4, -2)$ of γ_2 (see Figure 4).

$$\begin{pmatrix} -4 \\ +1 \\ +3 \\ +2 \end{pmatrix} \xrightarrow{\ b\ } \begin{pmatrix} -2 \\ -4 \\ +1 \\ +3 \end{pmatrix} \xrightarrow{\ b\ } \begin{pmatrix} -2 \\ -3 \\ -4 \\ +1 \end{pmatrix} \xrightarrow{\ a\ } \begin{pmatrix} -3 \\ -4 \\ -2 \\ +1 \end{pmatrix} \xrightarrow{\ b\ } \begin{pmatrix} -3 \\ -1 \\ -4 \\ -2 \end{pmatrix}$$
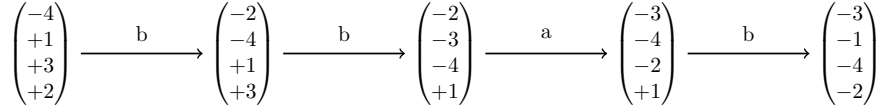
FIGURE 4. A path in $\mathcal{G}_4$ terminating at the start of the path in Figure 3.

The product matrix for the path γ_3 is

$$M_{\gamma_3}^{(4)} = \begin{pmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

Multiplying the vector $\mu/|\mu|$ by this matrix and normalizing gives the required length vector ξ. Thus we have constructed a 4-IET $T_3 := \mathcal{E}(\xi, r^{(0)})$ (see Figure 5) that has exactly one flip, and since it has a subinterval to which the Poincaré first return map is minimal and uniquely ergodic, is itself minimal and uniquely ergodic also. $\quad\square$
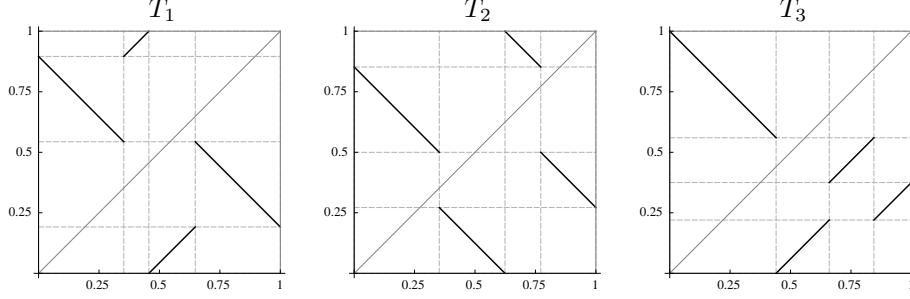
9. TRANSITIVE CETS WITH ARBITRARY NUMBER OF SUBINTERVALS AND FLIPS

In this section we provide a proof of the following result.

Theorem 9.1. $\mathcal{C}(n, f) \neq \emptyset$ for all $n \geq 4$ and $1 \leq f \leq n$.

Definition 9.2 (fake discontinuity). We say that an n-IET $T = \mathcal{E}(\lambda, \theta\pi)$ has a fake discontinuity if precisely one of the following cases happens:

(a) for exactly one $1 \leq i \leq n-1$, $\pi(i) = n$, $\pi(i+1) = 1$, and $\theta_i = \theta_{i+1} = +1$,

FIGURE 5. Graphs of the minimal IETs T_1, T_2, T_3.

(b) for exactly one $1 \leq i \leq n-1$, $\pi(i) = 1$, $\pi(i+1) = n$, and $\theta_i = \theta_{i+1} = -1$,
(c) $\pi(1) = \pi(n) + 1$ and $\theta_1 = \theta_n = +1$,
(d) $\pi(1) = \pi(n) - 1$ and $\theta_1 = \theta_n = -1$.

Henceforth, we call a transitive n-IET (respectively n-CET) with f flips an (n, f)-IET (respectively an (n, f)-CET). An (n, f)-IET without fake discontinuities is isomorphic to an (n, f)-CET whereas an (n, f)-IET with one fake discontinuity is isomorphic to either an $(n-1, f)$-CET or to an $(n-1, f-1)$-CET. Notice that an (n, n)-IET with one fake discontinuity is always isomorphic to an $(n-1, n-1)$-CET.

We shall need the following result.

Lemma 9.3. *There exist the following kind of transitive IETs:*
   (a) *A $(4, f)$-IET without fake discontinuities, for each $f = 1, 2, 3$;*
   (b) *A $(5, 5)$-IET with exactly one fake discontinuity;*
   (c) *A $(5, 5)$-IET without fake discontinuities.*

Proof. (a) Theorem 8.1 gives a (4,1)-IET without fake discontinuities. The permutation $p^{(2)} = (+3, -4, -2, +1)$ of γ_1 and $q^{(1)} = (-4, -1, +2, -3)$ of γ_2 provide, respectively, a $(4, 2)$-IET and a $(4, 3)$-IET, both without fake discontinuities. As for items (b) and (c), let $\gamma_b = \{p_b^{(i)}\}_{i=0}^{12}$ be the path starting at $(-3, -1, -5, -2, -4)$ having type itinerary $\{b, a, a, b, b, b, a, b, a, b, a, b, a\}$ and let $\gamma_c = \{p_c^{(j)}\}_{j=0}^{21}$ be the path starting at $(-2, -3, -4, -5, -1)$ having type itinerary

$$\{a, a, b, b, a, b, b, a, a, a, b, b, a, b, b, a, b, a, b, a, a, b\}.$$

We claim that these paths are special. Indeed, the fifth permutation of γ_b, $p_b^{(4)} = (4, -5, -1, -3, 2)$, belongs to a cycle whose product matrix B is eventually positive and the tenth permutation of γ_c, $p_c^{(9)} = (-4, 5, 1, -2, -3)$, belongs to a cycle whose product matrix C is eventually positive. We have

$$B = \begin{pmatrix} 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 3 & 4 & 3 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 2 & 2 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad C = \begin{pmatrix} 2 & 2 & 3 & 4 & 3 \\ 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 2 & 1 & 1 \\ 0 & 0 & 0 & 2 & 1 \\ 1 & 2 & 3 & 3 & 3 \end{pmatrix}.$$

To obtain the required IETs we proceed as in the previous sections. Let $\lambda_b \in \Delta_4$ and $\lambda_c \in \Delta_4$ be the Perron–Frobenius eigenvectors of B and C, respectively. Set

$\mu_b = M_{\gamma_b}^{(4)}\lambda_b$ and $\mu_c = M_{\gamma_c}^{(9)}\lambda_c$. The minimal 5-IET $T_b = \mathcal{E}(\mu_b/\mu_b, p_b^{(0)})$ has 5 flips and one fake discontinuity whereas the minimal $5-$IET $T_c = \mathcal{E}(\mu_c/\mu_c, p_c^{(0)})$ has 5 flips and no fake discontinuities. $\qquad\square$

Corollary 9.4. $\mathcal{C}(4, f) \neq \emptyset$, for all $f = 1, 2, 3, 4$.

Proof. It follows from itens (a) and (b) of Lemma 9.3. $\qquad\square$

Now we introduce two operators in the space of the interval exchange transformations that will allow us to construct IETs with arbitrary numbers of subintervals and flips. Given an (n, f)-IET $T : [0, 1] \rightarrow [0, 1]$, we can define an $(n + 1, f)$-IET $T_1 : [0, 2] \rightarrow [0, 2]$ and an $(n + 2, f + 2)$-IET $T_2 : [0, 3] \rightarrow [0, 3]$ such that the Poincaré first return map induced by T_1 (respectively T_2) on $[0, 1]$ is T. We set:

$$T_1(x) = \left\{ \begin{array}{ll} T(x) + 1 & \text{if } x \in [0, 1] \\ x - 1 & \text{if } x \in [1, 2] \end{array} \right. , \quad T_2(x) = \left\{ \begin{array}{ll} T(x) + 1 & \text{if } x \in [0, 1] \\ -x + 4 & \text{if } x \in [1, 2] \\ -x + 3 & \text{if } x \in [2, 3] \end{array} \right. .$$

Let $\rho : \Lambda_n \times \Sigma_n \rightarrow \Delta_{n-1} \times \Sigma_n$ be the projection map defined by $\rho(\lambda, \theta\,\pi) = (\lambda/|\lambda|, \theta\,\pi)$ and let α and β be the operators in the space of the transitive interval exchange transformations which at a (n, f)-IET $T : [0, 1] \rightarrow [0, 1]$ takes the value $\alpha(T) = \rho(T_1)$ and $\beta(T) = \rho(T_2)$, respectively. The operators α and β always produce IETs without fake discontinuities which so correspond to CETs with the same number of subintervals and flips.

We indicate now how to proceed in order to extend the results of this paper for more subintervals. The case of four subintervals was settled by the Corollary 9.4. The case of five subintervals can be approached in the following way. By Lemma 9.3, there exists a $(4, i)$-IET $E_i : [0, 1] \rightarrow [0, 1]$, $i = 1, 2, 3$. By Theorem 6.1, we have a $(4, 4)$-IET $E_4 : [0, 1] \rightarrow [0, 1]$ (with a fake discontinuity). The $(5, i)$-IETs $\alpha(E_i)$, $i = 1, 2, 3, 4$, have no fake discontinuities and so are isomorphic to $(5, i)$-CETs. The remaining case $(5, 5)$ is covered by Lemma 9.4. The case of $n > 5$ subintervals can be obtained by repeated applications of the operators α and β to our 4-IETs and 5-IETs examples, as it can be seen in the Figure 6 (a pair (n, f) means a uniquely ergodic transitive (n, f)-IET). Note that in Figure 6, the vertex $(4, 4)$ has a fake discontinuity, but all the other ones are free of fake discontinuities. We have proved Theorem 9.1.

Remark. *All the examples presented in this paper are uniquely ergodic. By applying the methods of Section 9 to the non-uniquely ergodic oriented IET example by Keane [8], one can easily construct transitive non-uniquely ergodic IETs with flips. However we leave open the problem of finding minimal non-uniquely ergodic IETs and CETs with flips.*
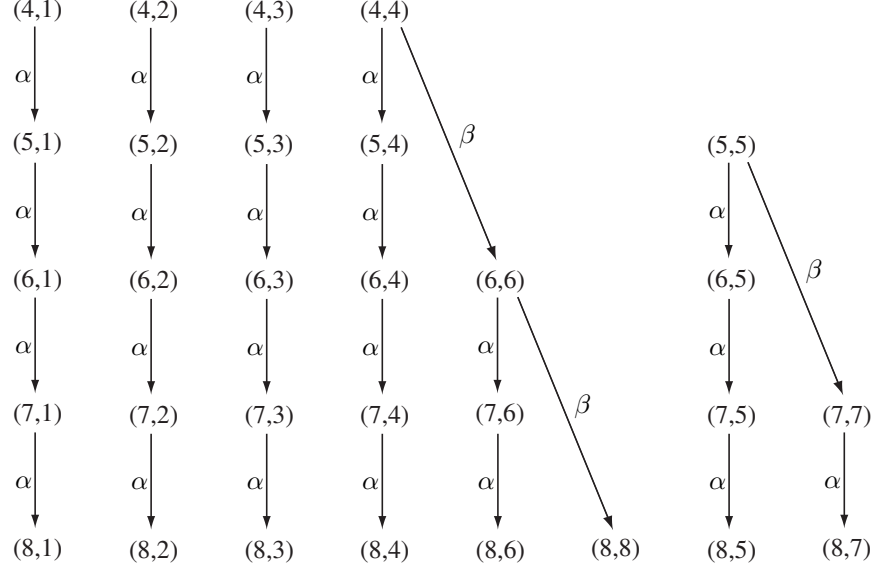
FIGURE 6. Tree of the (n, f)-IETs without fake discontinuities generated by α and β.

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